Math 53: Multivariable Calculus **Sections 102, 103**

Worksheet for 2020-04-27

Conceptual Review

$d\vec{S} = \pm \vec{r}_u \times \vec{r}_v \, du\, dv$

Question 1. When computing the flux of a vector field through a surface $\mathbf{r}(u, v)$, how might you decide whether to use $\mathbf{r}_u \times \mathbf{r}_v$ or $\mathbf{r}_v \times \mathbf{r}_u$? (How are these vectors related?)

$\vec{r}_u \times \vec{r}_v = -\vec{r}_v \times \vec{r}_u$

Compute one of them, say $\vec{r}_u \times \vec{r}_v$, plug in an easy choice of u, v, and see if it points in the correct direction. If yes, great! If not, take $\vec{r}_v \times \vec{r}_u$ instead, which is just $-\vec{r}_u \times \vec{r}_v$ (no need to take cross product again).

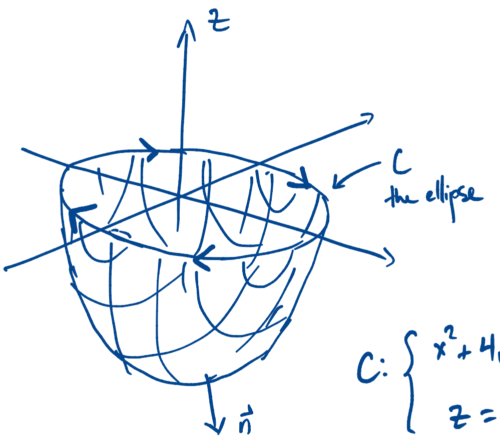
Problems

Problem 1. Let S be the surface $z = x^2 + 4y^2 - 4$, $z \le 0$ oriented downwards (i.e. negatively). Compute $\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$ where

$$\mathbf{F} = (y \log_2(x^2 + 4y^2 + z^2) + 3x^2y^2\cos(x^3))\mathbf{i} + (-3x + 2y\sin(x^3))\mathbf{j} + (e^{yz}\arctan(y^{x^2+1}))\mathbf{k}.$$

looks like Stokes!

elliptic paraboloid

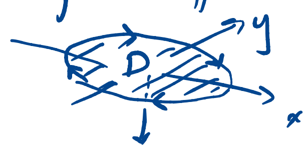


$$C: \begin{cases} x^2 + 4y^2 = 4 \\ z = 0 \end{cases}$$

(1) Use Stokes' to rewrite integral over C (what's the correct orientation of C?)

(2) Use the equation(s) of C to simplify the integrand

(3) Compute the integral, perhaps by another application of Stokes'/Green's Thm:



For a graph (like S), the two orientations are "up" and "down"

S up $\longleftrightarrow$ ∂S CCW viewed from above

S down $\longleftrightarrow$ ∂S CW viewed from above

$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} \overset{(1)}{=} \int_C \vec{F} \cdot d\vec{r}$$

$$\overset{(2)}{=} \int_C \langle \overbrace{2y + 3x^2y^2\cos(x^3)}^{P}, \overbrace{-3x + 2y\sin(x^3)}^{Q}, \sim\sim \rangle \cdot d\vec{r}$$

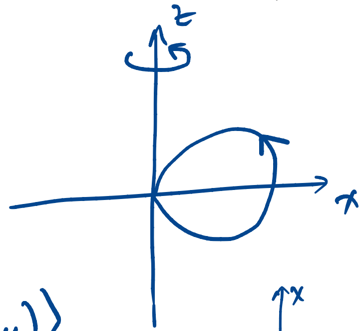
$\frac{dz}{dt}$ for curve is 0, so z-component of $\vec{F}$ is irrelevant.

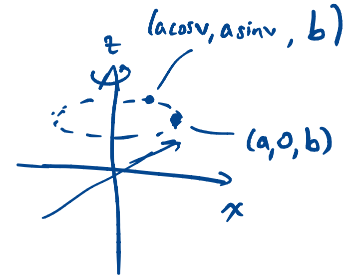
$$\overset{(3)}{=} \iint_D \langle \sim\sim, \sim\sim, Q_x - P_y \rangle \cdot \langle 0, 0, -1 \rangle \, dx\,dy = \iint_D 5 \, dx\,dy = \boxed{10\pi}$$

Problem 3. Use the divergence theorem to compute the volume enclosed by the surface obtained by rotating the curve $\langle \cos t, 0, \sin(2t)\rangle$ $(-\pi/2 \le t \le \pi/2)$ around the z-axis.

First need to parametrize.

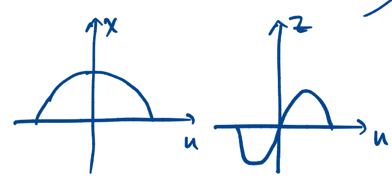
$\langle \cos u, 0, \sin(2u)\rangle$





$-\pi/2 \le u \le \pi/2 \qquad 0 \le v \le 2\pi$

$\vec{r}(u,v) = \langle \cos u \cos v, \cos u \sin v, \sin(2u)\rangle$



Volume: $\displaystyle\iiint_E 1\, dV \overset{\text{div thm}}{=} \iint_{\partial E} \vec{F} \cdot d\vec{S}$ as long as $\operatorname{div} \vec{F} = 1$.

Have many choices:

$\vec{F} = \langle x,0,0\rangle$ or $\langle 0,y,0\rangle$ or $\langle 0,0,z\rangle$ or $\frac{1}{3}\langle x,y,z\rangle$ or ......

By RHR: should take $\vec{r}_v \times \vec{r}_u$. (Alternatively, just take either one — if your final answer is negative then that means it should be the opposite since we are computing volume.)

$\parallel$

$\langle 2\cos u \cos v \cos(2u),$

$2\cos u \sin v \cos(2u),$

$\sin u \cos u\rangle$

Idea: compute $d\vec{S} = \pm\, \vec{r}_u \times \vec{r}_v\, du\, dv$ and pick $\vec{F}$ to make the dot product as simple as possible.

Looks like $\langle 0,0,z\rangle \cdot (\vec{r}_v \times \vec{r}_u)$ will be simplest.

$z = \sin(2u)$

$$\iint_{\partial E} \vec{F} \cdot (\vec{r}_v \times \vec{r}_u)\, du\, dv = \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin(2u) \overbrace{\sin u \cos u}^{(\sin 2u)/2}\, du\, dv$$

$$= \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1-\cos(4u)}{4}\, du\, dv = \boxed{\frac{\pi^2}{2}}$$